Suffolk County Community College

Michael J. Grant Campus

Department of Mathematics

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MAT 125
Pre-Calculus II

Final Exam: Solutions and Answers

Instructor:

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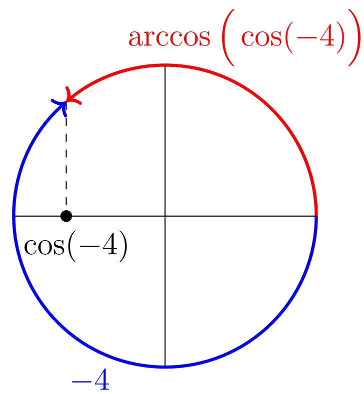
Problem 1. Consider the expression $\arccos\Big(\cos(-4)\Big)$.

(1). Draw -4, $\cos(-4)$ and $\arccos\Big(\cos(-4)\Big)$ in the same picture with the unit circle, showing how they are interconnected.

Space for your solution:



(2). Use the above picture to express $\arccos\Big(\cos(-4)\Big)$ without any trigonometric functions.

Space for your solution:

It is clear from the above picture that the length of the angle in question, when added to 4, should equal the length of the full circle 2π. Furthermore, given its positivity, it must be:

$$\arccos\Big(\cos(-4)\Big) = 2\pi - 4.$$

Problem 2. Solve the equation $\cos(2\theta) = 1 + \sin\theta$.

Space for your solution:

$\cos(2\theta) = 1 + \sin\theta \Leftarrow$ double angle formula for cos $\Rightarrow$ $1 - 2(\sin\theta)^2 = 1 + \sin\theta$

$\Leftarrow$ subtract 1 from both sides $\Rightarrow$ $-2(\sin\theta)^2 = \sin\theta \Leftarrow$ divide both sides by $\sin\theta$ $\Rightarrow$

$$\left[\begin{array}{l} \sin\theta = 0 \\ -2\sin\theta = 1 \end{array}\right. \Leftrightarrow \left[\begin{array}{l} \exists n \in \mathbb{Z}:\ \theta = \pi n \\ \sin\theta = -\frac{1}{2} \end{array}\right. \Leftrightarrow$$

$$\left[\begin{array}{l} \exists n \in \mathbb{Z}:\ \theta = \pi n \\ \exists n \in \mathbb{Z}:\ \theta = \arcsin(-\frac{1}{2}) + 2\pi n \\ \exists n \in \mathbb{Z}:\ \theta = \pi - \arcsin(-\frac{1}{2}) + 2\pi n \end{array}\right. \Leftrightarrow \left[\begin{array}{l} \exists n \in \mathbb{Z}:\ \theta = \pi n \\ \exists n \in \mathbb{Z}:\ \theta = -\frac{\pi}{6} + 2\pi n \\ \exists n \in \mathbb{Z}:\ \theta = \pi + \frac{\pi}{6} + 2\pi n \end{array}\right. \Leftrightarrow$$

$$\left[\begin{array}{l} \exists n \in \mathbb{Z}:\ \theta = \pi n \\ \\ \exists n \in \mathbb{Z}:\ \theta = -\frac{\pi}{6} + 2\pi n \\ \\ \exists n \in \mathbb{Z}:\ \theta = \frac{7\pi}{6} + 2\pi n \end{array}\right.$$

Problem 3. Solve the equation $\cot(t) = \sin(t)$.

Space for your solution:

$\cot(t) = \sin(t) \Leftarrow$ definition of $\cot(t)$ $\Rightarrow$ $\frac{\cos(t)}{\sin(t)} = \sin(t)$

$\Leftarrow$ multiply both sides by $\sin(t)$, but remember that it cannot be zero $\Rightarrow$

$$\left\{\begin{array}{l} \cos(t) = \big(\sin(t)\big)^2 \\ \sin(t) \neq 0 \end{array}\right. \Leftarrow \boxed{\text{Pythagorean identity}} \Rightarrow \left\{\begin{array}{l} \cos(t) = 1 - \big(\cos(t)\big)^2 \\ \sin(t) \neq 0 \end{array}\right. \Leftrightarrow$$

$$\left\{\begin{array}{l} \big(\cos(t)\big)^2 + \cos(t) - 1 = 0 \\ t \neq \pi \cdot n, n \in \mathbb{Z} \end{array}\right. \Leftarrow \boxed{\text{Quadratic formula}} \Rightarrow \left\{\begin{array}{l} \cos(t) = \frac{-1 \pm \sqrt{1^2 - 4 \cdot 1 \cdot (-1)}}{2 \cdot 1} \\ t \neq \pi \cdot n, n \in \mathbb{Z} \end{array}\right. \Leftrightarrow$$

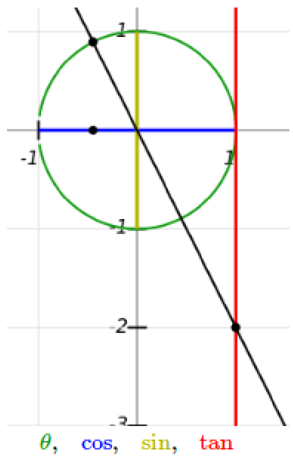
$$\left\{\begin{array}{l} \cos(t) = \frac{-1 \pm \sqrt{5}}{2} \\ t \neq \pi \cdot n, n \in \mathbb{Z} \end{array}\right. \Leftarrow \boxed{\text{Cosine cannot equal } \tfrac{-1-\sqrt{5}}{2} < -1.} \Rightarrow \left\{\begin{array}{l} \cos(t) = \frac{-1 + \sqrt{5}}{2} \\ t \neq \pi \cdot n, n \in \mathbb{Z} \end{array}\right. \Leftrightarrow$$

$t = \pm \arccos\left(\frac{-1+\sqrt{5}}{2}\right) + 2 \cdot \pi \cdot n, n \in \mathbb{Z}.$

Problem 4. In this problem, we will study $\cos\big(\arctan(y)\big)$.

(1). Suppose $\theta \in [0, \pi]$ and $\tan(\theta) = -2$. Draw -2, θ and $\cos(\theta)$ in the unit circle.

Space for your solution:



(2). Using the above picture, find $\cos(\theta)$.

Space for your solution:

By definition, $\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)}$. Since we know $\tan(\theta)$ and are trying to find $\cos(\theta)$, we should express $\sin(\theta)$ in terms of these two functions. Solving the Pythagorean identity $\big(\cos(\theta)\big)^2 + \big(\sin(\theta)\big)^2 = 1$ for the $\sin(\theta)$, we get $\sin(\theta) = \pm\sqrt{1 - \big((\cos(\theta)\big)^2}$. The given fact that $\theta \in [0, \pi]$ implies that the $\sin(\theta)$ is positive: $\sin(\theta) = \sqrt{1 - \big((\cos(\theta)\big)^2}$. Substituting this expression for $\sin(\theta)$ into the definition of $\tan(\theta)$, we get

$$\tan(\theta) = \frac{\sqrt{1 - \big((\cos(\theta)\big)^2}}{\cos(\theta)}.$$

Substituting the known value of $\tan(\theta) = -2$, we get the following equation for $\cos(\theta)$:

$$-2 = \frac{\sqrt{1 - \big((\cos(\theta)\big)^2}}{\cos(\theta)}$$

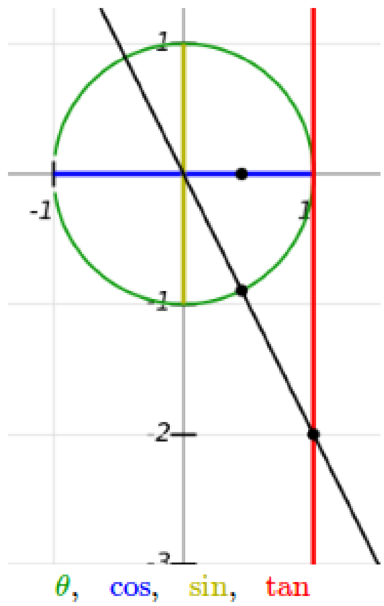
$$\Leftrightarrow \quad -2\cos(\theta) = \sqrt{1 - \big((\cos(\theta)\big)^2} \quad \Leftrightarrow \quad \begin{cases} 4\big(\cos(\theta)\big)^2 = 1 - \big((\cos(\theta)\big)^2 \\ \cos(\theta) < 0 \end{cases} \quad \Leftrightarrow$$

$$\begin{cases} 5\big(\cos(\theta)\big)^2 = 1 \\ \cos(\theta) < 0 \end{cases} \quad \Leftrightarrow \quad \begin{cases} \big(\cos(\theta)\big)^2 = \frac{1}{5} \\ \cos(\theta) < 0 \end{cases} \quad \Leftrightarrow \quad \cos(\theta) = -\sqrt{\frac{1}{5}} = -\frac{1}{\sqrt{5}}.$$

(3). Find $\cos\big(\arctan(-2)\big)$. More specifically, find an expression of this quantity that does not use any trigonometric functions. Can the work done for the previous sub-problem be used? To what extent?

Space for your solution:

The angle θ in the previous sub-problem was in the interval $[0, \pi]$, but $\arctan(-2)$ must be in the open interval $\big(-\frac{\pi}{2}, \frac{\pi}{2}\big)$. Thus the value of the cos we need here corresponds to the x-coordinate of the other point of intersection of the tilted line with the unit circle in the same picture as above:



The only difference is the sign: $\cos\big(\arctan(-2)\big) = \frac{1}{\sqrt{5}}$.

(4). For an arbitrary real number y, find $\cos\big(\arctan(y)\big)$. More specifically, find an expression of this quantity that does not use any trigonometric functions.

Space for your solution:

This is a generalization of the previous problem. Denote $x = \cos\big(\arctan(y)\big)$. Then, from the similarity of triangles in the above picture, we get the equation: $y = \pm\frac{\sqrt{1-x^2}}{x} \Leftrightarrow$ $y \cdot x = \pm\sqrt{1-x^2} \Leftrightarrow x^2 \cdot y^2 = 1 - x^2. \Leftrightarrow x^2(y^2+1) = 1 \Leftrightarrow x^2 = \frac{1}{y^2+1} \Leftrightarrow x = \pm\sqrt{\frac{1}{y^2+1}}$ $\Leftarrow$ Since $\arctan(y) \in \big(-\frac{\pi}{2}, \frac{\pi}{2}\big)$, we know that $x > 0$. $\Rightarrow x = \sqrt{\frac{1}{y^2+1}} = \frac{1}{\sqrt{y^2+1}}$.