Suffolk County Community College

Michael J. Grant Campus

Department of Mathematics

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MAT 142
Calculus with Analytic Geometry II

Final Exam: Solutions and Answers

Instructor:

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Problem 1. Compute the integral

$$\int \frac{\sin\left(\sqrt{x}\right)}{\sqrt{x}} \quad \mathtt{d}x.$$

Space for your solution:

$$\int \frac{\sin\left(\sqrt{x}\right)}{\sqrt{x}} \quad \mathtt{d}x = \boxed{\text{notice that reciprocal of } \sqrt{x} \text{ is almost the derivative of } \sqrt{x}} =$$

$$= 2\int \sin\left(\sqrt{x}\right)\frac{1}{2\sqrt{x}} \quad \mathtt{d}x = 2\int \sin\left(\sqrt{x}\right) \quad \mathtt{d}\sqrt{x}$$

$$= -2\int \left(-\sin\left(\sqrt{x}\right)\right) \quad \mathtt{d}\sqrt{x} = -2\int \mathtt{d}\left(\cos\left(\sqrt{x}\right)\right)$$

$$= -2\cos\left(\sqrt{x}\right) + C.$$

Problem 2. Compute the integral

$$\int \cos\left(\sqrt{x}\right) \quad \mathtt{d}x.$$

Space for your solution:

$$\int \cos\left(\sqrt{x}\right) \quad \mathtt{d}x = \boxed{u = \sqrt{x}} = \int \cos(u) \quad \mathtt{d}u^2 = \int 2u \ \cos(u) \quad \mathtt{d}u$$

$$= \boxed{\text{integration by parts}} = \int 2u \quad \mathtt{d}\sin(u) = 2u \ \sin(u) - \int \sin(u) \quad \mathtt{d}(2u)$$

$$= 2u \ \sin(u) - 2\int \sin(u) \quad \mathtt{d}u = 2u \ \sin(u) + 2\cos(u) + C$$

$$= 2\sqrt{x} \ \sin\left(\sqrt{x}\right) + 2\cos\left(\sqrt{x}\right) + C.$$

Problem 3. Find the

$$\int \frac{2x^3 - 5x^2 + 7}{x^2 - 4x + 4} \;\mathtt{d}x.$$

Space for your solution:

Performing the long division

$$\begin{array}{r|l} & 2x + 3 \\ \hline x^2 - 4x + 4 & 2x^3 - 5x^2 + 0x + 7 \\ & -(2x^3 - 8x^2 + 8x) \\ \hline & 3x^2 - 8x + 7 \\ & -(3x^2 - 12x + 12) \\ \hline & 4x - 5 \end{array}$$

we can find the integral

$$\int \frac{2x^3 - 5x^2 + 7}{x^2 - 4x + 4} \;\mathtt{d}x =$$

$$= \int \left(2x + 3 + \frac{4x - 5}{x^2 - 4x + 4}\right) \;\mathtt{d}x = \int 2x \;\mathtt{d}x + \int 3 \;\mathtt{d}x + \int \frac{4x - 5}{(x-2)^2} \;\mathtt{d}x$$

$$= \int \;\mathtt{d}x^2 + 3 \int \;\mathtt{d}x + A \int \frac{\mathtt{d}x}{x - 2} + B \int \frac{\mathtt{d}x}{(x-2)^2}$$

$$= x^2 + 3x + A \int \;\mathtt{d}x \;\ln\Big(\,|x - 2|\Big) + B \int \;\mathtt{d}x \;\frac{(x-2)^{-2+1}}{-2+1}$$

$$= x^2 + 3x + A \cdot \ln\Big(\,|x - 2|\Big) - \frac{B}{x - 2} + C,$$

where the coefficients A and B of partial fractions can be found from the identity holding for every x:

$$\forall x \in \mathbb{R} : A(x - 2) + B = 4x - 5$$

$$\Leftrightarrow \;\; \forall x \in \mathbb{R} : (A - 4)x + (2A + B + 5) = 0$$

$$\Leftrightarrow \;\; \begin{cases} A - 4 = 0 \\ -2A + B + 5 = 0 \end{cases} \quad \Leftrightarrow \;\; \begin{cases} A = 4 \\ B = 3, \end{cases}$$

yielding the final answer

$$\int \frac{2x^3 - 5x^2 + 7}{x^2 - 4x + 4} \;\mathtt{d}x = x^2 + 3x + 4 \cdot \ln\Big(\,|x - 2|\Big) - \frac{3}{x - 2} + C.$$

Problem 4. Consider the function $f(x) = \sin(x)$.

(1). Find the formula for

$$\left(\frac{\mathtt{d}}{\mathtt{d}x}\right)^i f(x)$$

with arbitrary x for $i = 0, 1, 2, 3, 4, 5$.

Space for your solution:

$$\left(\frac{\mathtt{d}}{\mathtt{d}x}\right)^0 f(x) = f(x) = \sin(x),$$

$$\left(\frac{\mathtt{d}}{\mathtt{d}x}\right)^1 f(x) = \Big(\sin(x)\Big)' = \cos(x),$$

$$\left(\frac{\mathtt{d}}{\mathtt{d}x}\right)^2 f(x) = \Big(\cos(x)\Big)' = -\sin(x),$$

$$\left(\frac{\mathtt{d}}{\mathtt{d}x}\right)^3 f(x) = \Big(-\sin(x)\Big)' = -\cos(x),$$

$$\left(\frac{\mathtt{d}}{\mathtt{d}x}\right)^4 f(x) = \Big(-\cos(x)\Big)' = \sin(x),$$

$$\left(\frac{\mathtt{d}}{\mathtt{d}x}\right)^5 f(x) \Big(\sin(x)\Big)' = \cos(x).$$

(2). Which point (or points) from the domain of the function f would be a good choice for the center of a Taylor polynomial for f, and why?

Space for your solution:

As we have seen in the previous sub-problem, for $f(x) = \sin(x)$, the expression needed for the Taylor polynomial:

$$\left(\frac{d}{d\,t}\right)^i \Bigg|_{t=x_0} f(t) = \pm\, t(x_0),$$

where the $t(x)$ itself is either $\cos(x)$ or $\sin(x)$. We can evaluate $t(x)$ when $x = 0$, which makes $x_0 = 0$ a good choice for the center of the Taylor polynomial of $f(x) = \sin(x)$.

(3). Explicate the Taylor polynomial approximation

$$f(x) = \sum_{i=0}^{n} \frac{1}{i\,!} \cdot \left(\left(\frac{d}{d\,t}\right)^{i} \Bigg|_{t=x_0} f(t) \right) \cdot (x - x_0)^i + \frac{1}{n\,!} \int_{t=x_0}^{x} \left(\left(\frac{d}{d\,t}\right)^{n+1} f(t) \right) \cdot (x-t)^n \; d\,t$$

for $f(x) = \sin(x)$, $n = 5$, and x_0 selected in the previous sub-problem. The final answer must contain neither the symbols of differentiation, nor the sigma notation.

Space for your solution:

$$\sin(x) =$$

$$= \sum_{i=0}^{n} \frac{1}{i\,!} \cdot \left(\left(\frac{d}{d\,t}\right)^{i} \Bigg|_{t=0} \sin(t) \right) \cdot x^i + \frac{1}{n\,!} \int_{t=0}^{x} \left(\left(\frac{d}{d\,t}\right)^{n+1} \sin(t) \right) \cdot (x-t)^n \; d\,t$$

$$= \frac{1}{0!} \cdot \sin(0) \cdot x^0 + \frac{1}{1!} \cdot \cos(0) \cdot x^1 - \frac{1}{2!} \cdot \sin(0) \cdot x^2 - \frac{1}{3!} \cdot \cos(0) \cdot x^3 + \frac{1}{4!} \cdot \sin(0) \cdot x^4 + \frac{1}{5!} \cdot \cos(0) \cdot x^5$$

$$+ \frac{1}{5\,!} \int_{t=0}^{x} \left(\left(\frac{d}{d\,t}\right)^{5+1} \sin(t) \right) \cdot (x-t)^5 \; d\,t$$

$$= 0 + x - 0 - \frac{1}{6}\; x^3 + 0 + \frac{1}{120}\; x^5 - \frac{1}{120} \int_{t=0}^{x} \sin(t) \cdot (x-t)^5 \; d\,t$$

$$= x - \frac{1}{6}\; x^3 + \frac{1}{120}\; x^5 - \frac{1}{120} \int_{t=0}^{x} \sin(t) \cdot (x-t)^5 \; d\,t.$$

(4). Explicate the Taylor polynomial approximation

$$f(x) = \sum_{i=0}^{n} \frac{1}{i\,!} \cdot \left(\left(\frac{d}{d\,t}\right)^{i} \Bigg|_{t=x_0} f(t) \right) \cdot (x - x_0)^i + \frac{1}{n\,!} \int_{t=x_0}^{x} \left(\left(\frac{d}{d\,t}\right)^{n+1} f(t) \right) \cdot (x-t)^n \; d\,t$$

for $f(x) = \sin(x)$ and arbitrary n at the x_0 selected previously. The final answer must contain the sigma notation, but may have the differentiation symbol only in the error term.

Space for your solution:

For the odd $n = 2k+1$, we can get the following expression, denoting $i = 2j+1$:

$$\sin(x) =$$

$$= \sum_{j=0}^{k} \frac{(-1)^j}{(2j+1)\,!} \cdot x^{2j+1} + \frac{1}{(2k+1)\,!} \int_{t=0}^{x} \left(\left(\frac{d}{d\,t}\right)^{2(k+1)} \sin(t) \right) \cdot (x-t)^{2k+1} \; d\,t.$$

The Taylor polynomial of even degree $n = 2k$ is the same as that of the odd degree $2k-1$.

(5). Explicate the Taylor polynomial approximation

$$f(x) = \sum_{i=0}^{n} \frac{1}{i\ !} \cdot \left(\left(\frac{d}{d\ t}\right)^{i} \Bigg|_{t=x_0} f(t) \right) \cdot (x - x_0)^i + \frac{1}{n\ !} \int_{t=x_0}^{x} \left(\left(\frac{d}{d\ t}\right)^{n+1} f(t) \right) \cdot (x-t)^n \ d\ t$$

for $f(x) = \sin(x)$ and $n = 1000$ at the x_0 selected previously. The final answer must contain the sigma notation, but may have the differentiation symbol only in the error term.

Space for your solution:

This particular value of n is even, so we need to compute the Taylor polynomial for the previous value of $n = 999 = 2 \cdot 499 + 1$, making the required value of $k = 499$:

$$\sin(x) \ = \ \sum_{j=0}^{499} \frac{(-1)^j}{(2j+1)\ !} \cdot x^{2j+1} + \frac{1}{999\ !} \int_{t=0}^{x} \left(\left(\frac{d}{d\ t}\right)^{1000} \sin(t) \right) \cdot (x \ - \ t)^{999} \ \ d \ \ t.$$

(6). For the degree n Taylor polynomial approximation of $\sin(x)$, find a computable estimate of the error that has the limit 0 as $n \to +\infty$.

Space for your solution:

The absolute value of the error of the Taylor polynomial approximation can be estimated from above as follows:

$$\left| \frac{1}{n\ !} \int_{t=0}^{x} \left(\left(\frac{d}{d\ t}\right)^{n+1} \sin(t) \right) \cdot (x-t)^n \ d\ t \right|$$

$$\leq \left| \frac{1}{n\ !} \int_{t=0}^{x} \max_{t \in [0,x]} \left(\left| \left(\frac{d}{d\ t}\right)^{n+1} \sin(t) \right| \right) \cdot (x-t)^n \ d\ t \right|$$

$$= \frac{1}{n\ !} \cdot \max_{t \in [0,x]} \left(\left| \left(\frac{d}{d\ t}\right)^{n+1} \sin(t) \right| \right) \cdot \left| \int_{t=0}^{x} (x-t)^n \ d\ t \right|$$

$$\leq \frac{1}{n\ !} \cdot 1 \cdot \left| \frac{x^{n+1}}{n+1} \right| = \frac{|x|^{n+1}}{(n+1)\ !}.$$

(7). Estimate $\sin(1)$ with guaranteed precision $\varepsilon = 0.01$.

Space for your solution:

The previously found estimate

$$\frac{|x|^{n+1}}{(n+1)\,!} = \frac{1}{(n+1)\,!}$$

of the error of Taylor approximation of $\sin(1)$ is less than the $\varepsilon = 0.01$ if $n = 4$:

$$\frac{1}{(4+1)\,!} = \frac{1}{5\,!} = \frac{1}{120} < \frac{1}{100} = 0.01.$$

Since the Taylor polynomial of degree 4 is the same as that of degree 3, the Taylor polynomial of degree 3

$$x - \frac{1}{6}\,x^3$$

already gives the required precision:

$$\sin(1) \approx 1 - \frac{1}{6} = 5/6.$$